



TITLE:

Variable Coefficients A-stable Explicit Runge-Kutta Methods (II)

AUTHOR(S):

Nakashima, Masaharu

CITATION:

Nakashima, Masaharu. Variable Coefficients A-stable Explicit Runge-Kutta Methods (II). 数理解析研究所講究録 1997, 990: 165-168

ISSUE DATE:

1997-04

URL:

<http://hdl.handle.net/2433/61090>

RIGHT:

Variable Coefficients A-stable Explicit Runge-Kutta Methods (II)

By Masaharu Nakashima (中島正治)

Department of Mathematics, Faculty of Science, Kagoshima University
Korimoto cho 21-35, Kagoshima city 890, Japan
e-mail; naka @sci.kagoshima-u.ac.jp

§1. Introduction

We study the numerical method for solving the stiff initial value problem

$$\dot{y} = f(x, y), y(x_0) = y_0.$$

The method which we propose is the variable coefficients Runge-Kutta methods (abbr; R-K methods);

$$\begin{aligned} y_{n+1} &= y_n + h \sum_{i=1}^r b_i k_i, \\ k_1 &= f(x_n, y_n), \\ k_i &= f(x_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} k_j), \\ c_i &= \sum_{j=1}^{i-1} a_{ij}, \quad (i = 2, \dots, r). \end{aligned} \quad (1.1)$$

§2. Derivation of Methods for a single equation of the first order

In this section, we consider two-stage first order Runge-Kutta formulae.

1-st order condition of (1.1) is

$$\sum_i b_i = 1. \quad (2.1)$$

Let us now apply the 2-stage first order R-K methods (1.1) to the test function

$$\dot{y} = \lambda y, \quad \operatorname{Re}(\lambda) < 0, \quad (2.2)$$

we have

$$y_{n+1} = (1 + z + b_2 a_{21} z^2) y_n, \quad (z = h\lambda). \quad (2.3)$$

If we set the coefficient $b_2 a_{21}$ in the form

$$b_2 a_{21} = \frac{1}{1 - z + z^2}, \quad (2.4)$$

then (2.3) reduces to

$$y_{n+1} = \frac{1}{1 - z + z^2} y_n. \quad (2.5)$$

which is an A-stable algorithm.

In determining the coefficients from the order condition (2.1), we set b_2 as a free parameter defined in the form

$$b_2 a_{21} = \frac{y_n}{y_n - 3hk_1 + 2hk_2}. \quad (2.6)$$

It is easily seen that the value of (2.6) applied to the test function (2.2) is the same as that of (2.4). Solving order condition (2.1) with (2.6), we have the coefficients

$$b_2 = \frac{y_n}{a_{21}(y_n - 3hk_1 + 2hk_2)}, \quad b_1 = 1 - b_2, \quad a_{21} : \text{free parameter}. \quad (2.7)$$

§3. Numerical integration for s-systems of equations of the first order

We consider numerical integration for s-systems differential equation;

$$\dot{Y} = F(Y), \quad (3.1)$$

with

$$Y = ({}^1y, {}^2y, \dots, {}^sy), F(Y) = ({}^1f(Y), {}^2f(Y), \dots, {}^sf(Y)).$$

We consider r-stage explicit R-K methods for s-systems equation (3.1)

$$\begin{aligned} {}^l y_{(n+1)} &= {}^l y_n + h \sum_{i=1}^r b_i {}^l k_i, \\ {}^l k_1 &= {}^l f(x_n, {}^1 y_n, {}^2 y_n, \dots, {}^s y_n), \\ {}^l k_i &= {}^l f(x_n + c_i h, {}^1 y_n + \sum_{j=1}^{j-1} a_{ij} {}^1 k_j, \dots, {}^q y_n + \sum_{j=1}^{j-1} a_{ij} {}^q k_j), \quad (l = 1, 2, \dots, s), \end{aligned} \quad (3.2)$$

${}^l b_i$: varies in each steps and with the component-wise.

Introducing the vector notations

$$Y_n = ({}^1 y_n, {}^2 y_n, \dots, {}^s y_n), B_i = \text{diag}[{}^1 b_i, {}^2 b_i, \dots, {}^s b_i], K_i = [{}^1 k_i, {}^2 k_i, \dots, {}^s k_i] \quad (i = 1, 2, \dots, s-1),$$

we may write (3.2) in the form;

$$Y_{n+1} = Y_n + \sum_{i=1}^{s-1} B_i K_i. \quad (3.3)$$

As the same reason stated in §2, we may set the coefficients ${}^i b_2 a_{21}$ of i -th component in the form

$${}^i b_2 a_{21} = \frac{{}^i y_n}{{}^i y_n - 3h {}^i k_1 + 2h {}^i k_2}. \quad (3.4)$$

From order condition (2.1) with (3.4), we have the coefficients of i -th component in the forms

$${}^i b_2 a_{21} = \frac{{}^i y_n}{{}^i y_n - 3h {}^i k_1 + 2h {}^i k_2}, \quad {}^i b_1 = 1 - {}^i b_2, \quad a_{21}; \text{ free parameter.} \quad (3.5)$$

The stability results for system equation based on the test function (2.2) can not to be watertight when applied to the variable coefficient methods, so it is essential to use $\dot{Y} = AY$ as test function, and not $\dot{y} = \lambda y$. We state by attacking the simple matrix A defined by

$$\dot{Z} = \Lambda Z \quad \text{with} \quad \Lambda = HAH^{-1} \quad \text{for some } H \quad \text{and} \quad \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_s\}. \quad (3.6)$$

$$\begin{array}{ccc} \dot{Y} = AY & \xrightarrow[\text{numerical integration}]{Z=HY} & \dot{Z} = \Lambda Z \\ \downarrow Y_n & & \downarrow Z_n \end{array}$$

Clearly the numerical processes $\{Z_n\}$ derived by (3.2) with (3.5) is stable on the above diagram, the transformation $\tilde{Z}_n = HY_n$ does not equivalent to the numerical processes Z_n . We study the stability of numerical processes \tilde{Z}_n on the most simple differential equation:

$$\dot{Y} = AY \quad \text{with} \quad A = \begin{pmatrix} p & q \\ q & p \end{pmatrix}, \quad (p+q < 0, p-q < 0). \quad (3.7)$$

Integrating the differential equation (3.7), we have

$$Y_{n+1} = (I + hA + h^2 CA^2)Y_n, \quad (3.8)$$

with

$$\begin{aligned} {}^1y_{n+1} &= {}^1y_n + h(p {}^1y_n + q {}^2y_n) + h^2({}^1b_2a_{21}(p {}^1f_n + q {}^2f_n)), \\ {}^2y_{n+1} &= {}^2y_n + h(q {}^1y_n + p {}^2y_n) + h^2({}^2b_2a_{21}(q {}^1f_n + p {}^2f_n)), \\ {}^1f_n &= p {}^1y_n + q {}^2y_n, {}^2f_n = q {}^1y_n + p {}^2y_n, \\ {}^1b_2a_{21} &= \frac{{}^1y_n}{({}^1y_n - 3h {}^1k_1 + 2h {}^1k_2)}, \\ {}^2b_2a_{21} &= \frac{{}^2y_n}{({}^2y_n - 3h {}^2k_1 + 2h {}^2k_2)}, \end{aligned}$$

From (3.8), we have

$$\begin{aligned} \tilde{Z}_{n+1} &= \{I + h\Lambda + h^2(I - h\Lambda + (h\Lambda)^2)^{-1}\Lambda^2\}\tilde{Z}_n \\ &\quad + h^2H\{C - (I - hA + (hA)^2)\}A^2H^{-1}\tilde{Z}_n, \end{aligned} \quad (3.9)$$

Setting K by

$$\begin{aligned} K &= I + h\Lambda + h^2\{I - h\Lambda + (h\Lambda)^2\}^{-1} \\ &= \begin{pmatrix} 1 - \lambda_2 + \lambda_2^2 & 0 \\ 0 & 1 - \lambda_1 + \lambda_1^2 \end{pmatrix}^{-1}, \\ \lambda_1 &= h(p + q), \lambda_2 = h(p - q) \end{aligned}$$

and Q_n by

$$\begin{aligned} Q_n &= H\{C - (I - hA + (hA)^2)\}^{-1}A^2H^{-1} \\ &= \frac{1}{2} \begin{pmatrix} (s_1 + s_2) & -(s_2 - s_1) \\ -(s_2 - s_1) & (s_1 + s_2) \end{pmatrix} - (I - h\Lambda + (h\Lambda)^2)^{-1}\Lambda^2, \\ (s_1 &= {}^1b_2a_{21}, s_2 = {}^2b_2a_{21}). \end{aligned}$$

We may write (3.9) in the form

$$\tilde{Z}_{n+1} = (K + Q_n)\tilde{Z}_n, \quad \tilde{Z}_1 = \Theta(\tau), \quad (n = 1, 2, \dots). \quad (3.10)$$

Using the standard thchnique, we obtain the following Lemmas.

Lemma 1

$$\tilde{Z}_{n+1} = K^n\Theta(\tau) + \sum_{\nu=0}^{n-1} K^{n-(\nu+1)}Q_\nu\tilde{Z}_\nu.$$

Lemma 2

$$\tilde{Z}_{n+1} = K^n\Theta(\tau) + \Pi_{\nu=0}^{n-1}S_{n-\nu}Q_0\Theta(\tau) + \sum_{\mu=1}^n(\Pi_{\nu=0}^nS_{n-\nu})(\Pi_{\nu=0}^\mu S_\nu^{-1})Q_\mu K^\mu$$

with $S_\nu = K + Q_\nu$,

where $S_\nu = S_\nu(r_1, r_2)$ and $S_\nu(r_1, r_2) = K + Q(r_1(\nu), r_2(\nu))$ with

$$Q_\nu(r_1(\nu), r_2(\nu)) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

$$c_{11} = p_1 + \frac{1}{1 - \lambda_1 + \lambda_1^2}, \quad c_{22} = p_2 + \frac{1}{1 - \lambda_2 + \lambda_2^2},$$

$$c_{12} = c_{21} = \frac{1}{2}(p_1 - p_2),$$

with $p_1 = -h * (p + qr_1) / \{1 - h(p + qr_1) + h^2(p^2 + q^2) + 2pqh^2r_1\}$,

$$p_2 = -h * (p + qr_2) / \{1 - h(p + qr_2) + h^2(p^2 + q^2) + 2pqh^2r_2\}, r_1(\nu) = {}^2y_\nu / {}^1y_\nu, r_2(\nu) = {}^1y_\nu / {}^2y_\nu \quad (3.11)$$

Using Lemma [1,2], we have

Theorem 1

If there exists the constant T such that $\|\Pi_{l=m}^n S(r_1(l), r_2(l))\| \leq T$ for all $m, n (m \leq n)$, then $\{\tilde{Z}_n\}$ stable.

Numerical Example

Setting $r_1 = r_2 = 1$ or $r_1 = r_2 = -1$ in (3.11), we compute the following differential equation $\dot{Y} = AY$ with

$$A = \begin{pmatrix} -550.5 & q \\ q & -500.5 \end{pmatrix}, Y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

The datas are the absolute error of numerical solution with step size $h = 1$

x		1	3	
q=0	1y_n	0	0	
	2y_n	0.796E-5	0,126E-6	
q=1	1y_n	0.210E+1	0.158E-4	
	2y_n	0.210E+1	0.158E-4	
x		1	3	5
q=50	1y_n	0.121E+3	0.487E-1	0.195E-4
	2y_n	0.121E+3	0.487E-1	0.195E-4
q=499.5	1y_n	0.332E+7	0.366E+6	0.540E+1
	2y_n	0.332E+7	0.366E+6	0.540E+1

$$\lambda_1 = -500.5 + q, \lambda_2 = -500.5 - q.$$

References

- [1] M. Nakashima, Variable Coefficients A-stable Explicit Runge-Kutta Methods
- [2] P.j. van der Houwen, Construction of Integration Formulas for Initial Value Problem.